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RIEMANN WAVES IN AN ELASTIC MEDIUM WITH SMALL CUBIC ANISOTROPY†

E. I. SVESHNIKOVA

Moscow

e-mail: sveshn@mech.math.msu.su; kulik@mi.ras.ru

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Riemann waves in a weakly non-linear weakly anisotropic elastic material possessing the property of cubic symmetry are considered. The elastic potential is taken in the form of a series expansion in powers of the strain up to the fourth order of smallness. Anisotropy is represented in this expansion by cubic terms with a small coefficient. With that model, a solution is obtained and investigated in the form of quasi-periodic Riemann waves propagating along the principal diagonal of a cube. The characteristic velocities are found, the integral curves on the phase plane are constructed, and the direction in which the parameters vary along the integral curves, resulting in inversion of the solution profile, is indicated. © 2005 Elsevier Ltd. All rights reserved.

1. STATEMENT OF THE PROBLEM. SPECIFICATION OF THE ELASTIC POTENTIAL

In an elastic medium with elastic potential Φ, continuous solutions of the equations of motion are sought in the form of plane Riemann waves. The investigation will be carried out in Lagrangian variables in the Cartesian system of the initial state. The axis $x_3 = x$ is orthogonal to the wave front and the x_1, x_2 axes lie in the plane of the front. Strains are characterized by the components $\partial w_i/\partial x_k$ $(i, k = 1, 2, 3)$ of the gradient of the displacement vector $\mathbf{w}$ and are assumed to be small, $\sim\varepsilon$. In a plane wave, only the components $\partial w_i/\partial x = u_i(x, t)$ vary; the other components $\partial w_i/\partial x_\alpha$, $(\alpha = 1, 2)$ are constant and taken to be equal to zero.

The equations of motion are

$$\rho_0\frac{\partial \upsilon_i}{\partial t} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial u_i}, \quad \frac{\partial \upsilon_i}{\partial x} = \frac{\partial u_i}{\partial t}; \quad i = 1, 2, 3 \tag{1.1}$$

where $\upsilon_i = \partial w_i/\partial t$ are the components of the velocity vector, $\Phi = \rho_0 U$ is the elastic potential, where U is the internal energy per unit mass and ρ_0 is the density of the medium, which, assuming homogeneity, is constant and will henceforth be taken to be equal to unity. The entropy in a Riemann wave is assumed to be constant, so that $\Phi = \Phi(u_i)$. Since the strains are small, the function Φ may be represented by a series in powers of u_i, retaining as many powers of u_i as necessary for the effects of the non-linearity of the medium to appear. It is well known [1, 2] that to that end it is sufficient to retain fourth powers of u_i. Because of the non-linearity, purely longitudinal and purely transverse elastic waves become quasi-longitudinal and quasi-transverse. In what follows, as more interesting, we will consider only quasi-transverse waves in which the variation of the longitudinal component u_3 is one order of magnitude less than u_1 and u_2. This will be taken into consideration in the series expansion of Φ. For quasi-transverse non-linear waves in an isotropic medium, the essential terms of the expansion of Φ turn out to be

$$\Phi_{is} = \Phi_0 = \frac{f_0}{2}(u_1^2 + u_2^2) + \frac{d}{2}u_3^2 + bu_3(u_1^2 + u_2^2) + \frac{h}{4}(u_1^2 + u_2^2)^2 \tag{1.2}$$

where f, b, d and h are the constants of elasticity of the medium.

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It was shown in [2] that quasi-transverse waves in a medium with small anisotropy (which is always the case in practice) acquire new qualitative properties. For the effects of non-linearity and anisotropy to appear in their interactions, the terms in the expansion of Φ that account for anisotropy must be of the same order as the non-linear terms, i.e. of the fourth order in u_i. If the aforementioned properties of the medium are represented by terms of different orders, the effect of one of them will suppress the other. In the case of anisotropy of a general type, the next term in the expansion (1.2) has the form $g(u_2^2 - u_1^2)$, where $g \sim \varepsilon^2$. This term appears even naturally in an isotropic medium when there is constant prestrain in the plane of the wave front. Non-linear waves in such a medium have been studied in numerous publications, including [2]; Riemann waves were already described in [3].

However, more interest has always been aroused by the appearance of small anisotropy of another type. In a medium without prestrain, for example, it may happen that the expansion of the function Φ contains no terms quadratic in u_i, and the anisotropy may than be represented by cubic terms in u_i with a small coefficient $g \sim \varepsilon$. It turns out that under certain conditions the elastic potential of a medium possessing cubic symmetry has such properties. The elastic potential of a cubic crystal has been represented [4] by an expansion of up to third powers in the components ε_{ij} of the Green strain tensor relative to axes attached to the axes of symmetry of the crystal. For further analysis, one must transform its expression to new axes, rotated through a suitable angle, so that the x_3 axis points along the direction of propagation of the plane wave, and the strains ε_{ij} themselves are expressed in terms of the components u_i. Such a representation has been accomplished [5] for three special directions of motion of the wave:

along an edge of the cube

$$\Phi_{\text{cub}}^{(r)} = \alpha(u_1^2 + u_2^2) + \beta u_3^2 + Au_3^3 + Bu_3(u_1^2 + u_2^2)$$

along the diagonal of a face

$$\Phi_{\text{cub}}^{(d)} = \alpha_1(u_1^2 + u_2^2) + \alpha_2(u_1^2 - u_2^2) + \beta u_3^2 + Au_3^3 + u_3(B_1u_1^2 + B_2u_2^2)$$

along the principal diagonal of the cube

$$\Phi_{\text{cub}}^{(dm)} = \alpha(u_1^2 + u_2^2) + \beta u_3^2 + A(3u_1^2u_2 - u_2^3) + u_3(B_1u_1^2 + B_2u_2^2 + B_3u_3^2)$$

It is obvious that the function $\Phi_{\text{cub}}^{(r)}$ contains only isotropic terms and is of no interest in the problem under consideration here. In the expression for $\Phi_{\text{cub}}^{(d)}$ the terms $\alpha_2(u_1^2 - u_2^2)$ representing the anisotropy have the same form as in [2, 3]. Attention will be confined from now on, therefore, to the case of wave motion along the principal diagonal of the cube. In that case, the last two terms will be included in the function $\Phi_{is} = \Phi_0$. Since the anisotropy is assumed to be small, the additional term in the general expansion of Φ, namely, $\Phi_{\text{cub}}^{(dm)}$, must occur with a small coefficient $g \sim \varepsilon$. Taking into consideration that for quasi-transverse waves $u_3 \sim \varepsilon^2$, only the first of the cubic terms need be retained in the expression for $\Phi_{\text{cub}}^{(dm)}$. As a result, the elastic potential for quasi-transverse waves with small cubic anisotropy is

$$\Phi = \Phi_0 + g(3u_1u_2^2 - u_1^3) \tag{1.3}$$

Note that if the waves propagate in a direction slightly different from that indicated, new terms may appear in the elastic potential, in particular, anisotropic terms quadratic in u_i with small coefficients. If the deviations in the direction of propagation of the wave are sufficiently small, one can use the representation of Φ given by formula (1.3).

It was shown in [2] that, for non-linear quasi-transverse waves of small amplitude, the variation of the longitudinal component u_3 may be expressed, using Eqs (1.1), in terms of the variations of the transverse components u_1 and u_2, after which the elastic potential can be written as function of the transverse components u_1 and u_2 only. Since the additional anisotropic term in formula (1.3) does not contain u_3, the aforementioned property of the function Φ remains the same for a cubic crystal as well. In quasi-transverse waves, therefore, the elastic potential Φ may be replaced by a function $H(u_1, u_2)$ of only two variables, which may be treated as the elastic potential of an equivalent incompressible medium in which two purely transverse waves are propagating. The function $H(u_1, u_2)$ has the form

$$H(u_1, u_2) = \frac{f}{2}(u_1^2 + u_2^2) - \frac{\kappa}{4}(u_1^2 + u_2^2)^2 + g(3u_1u_2^2 - u_1^3) \tag{1.4}$$

where f, κ and g are constants of elasticity. The coefficient f (which also included the isotropic terms of the function $\Phi_{\text{cub}}^{(dm)}$) has the meaning of the squared velocity of a linear transverse wave in an isotropic medium. The coefficient $\kappa = h - 2b^2/(d - f_0)$ serves as a non-linearity parameter, on whose sign the behaviour of the waves depends. The coefficient $g \sim \varepsilon$ represents an anisotropy parameter which, by altering the numbering of the axes, may always be made positive.

It is readily shown that the function $H(u_1, u_2)$ given by formula (1.4) possesses symmetry: If the u_1 and u_2 axes are rotated as a whole through on angle of $2\pi/3$, its form remains unchanged relative to the new variables. In addition, it is symmetrical about the u_1 axis.

Thus, the system of equations (1.1) now has the form

$$\frac{\partial \upsilon_\alpha}{\partial t} = H_{\alpha\beta}\frac{\partial u_\beta}{\partial x}, \quad \frac{\partial u_\alpha}{\partial t} = \frac{\partial \upsilon_\alpha}{\partial x}, \quad H_{\alpha\beta} = \frac{\partial^2 H}{\partial u_\alpha \partial u_\beta}, \quad \alpha, \beta = 1, 2 \tag{1.5}$$

where the function $H(u_1, u_2)$ is given by formula (1.4) and it is assumed that $\rho_0 = 1$.

2. THE CHARACTERISTIC VELOCITIES AND INTEGRAL CURVES OF RIEMANN WAVES

We will seek a solution of system (1.5) in the form $u_\alpha = u_\alpha(\theta(x, t))$, $\upsilon_\alpha = \upsilon_\alpha(\theta(x, t))$, where $\theta(x, t)$ is some function satisfying the equation

$$\frac{\partial \theta}{\partial t} + c(\theta)\frac{\partial \theta}{\partial x} = 0$$

This solution represents a Riemann wave. System (1.5) becomes a system of ordinary differential equations in $du_\alpha/d\theta$:

$$\begin{aligned} (H_{11} - c^2)\frac{du_1}{d\theta} + H_{12}\frac{du_2}{d\theta} &= 0 \\ H_{12}\frac{du_1}{d\theta} + (H_{22} - c^2)\frac{du_2}{d\theta} &= 0 \end{aligned} \tag{2.1}$$

The system has a non-trivial solution if $|H_{\alpha\beta} - \delta_{\alpha\beta}\lambda| = 0$. The roots of this equation are the eigenvalues λ_1 and λ_2 of the matrix $\|H_{\alpha\beta}\|$ (they are identical with the squared velocities of the Riemann waves: $\lambda_\alpha = c_\alpha^2$), and its eigenvector determines at each point u_1, u_2 of the phase plane the direction of the integral curves (ICs) of the desired solution.

The characteristic velocities are computed from the formulae

$$\begin{aligned} \lambda_{1,2} &= c_{1,2}^2 = f - 2\kappa(u_1^2 + u_2^2) \mp |\kappa| Q \\ Q &= \sqrt{(u_1^2 - u_2^2 + 2Gu_1)^2 + 4u_2^2(u_1 - G)^2}, \quad G = g/\kappa \end{aligned} \tag{2.2}$$

The sign of the root in formula (2.2) will be taken in such a way that $c_1 \le c_2$. The Riemann waves corresponding to characteristic velocity c_1 will be called slow, and the others, corresponding to c_2, fast. The choice of sign in formula (2.2), hence also the behaviour of the solution, depends on the sign of κ. To fix our ideas, all further reasoning will be carried out for $\kappa > 0$ and the upper sign in formula (2.2) will be taken for slow waves (c_1), the lower sign for fast waves (c_2). For media with $\kappa < 0$ the investigation may be carried out in analogous fashion; the results will be given at the end of Section 4.

System (2.1) gives equations for the directions of two families of ICs, which are orthogonal because of the symmetry of the matrix $|H_{\alpha\beta}\|$:

$$\frac{du_2}{du_1} = \frac{\lambda - H_{11}}{H_{12}} = \frac{u_1^2 - u_2^2 - 2Gu_1 \pm Q}{2u_2(u_1 - G)} \tag{2.3}$$

This equation depends on the parameters of non-linearity κ and anisotropy g only through their quotient $G = g/\kappa$. One can thus change scale on the u_1 and u_2 axes, that is, define $u_1 = u_1'G$, $u_2 = u_2'G$,

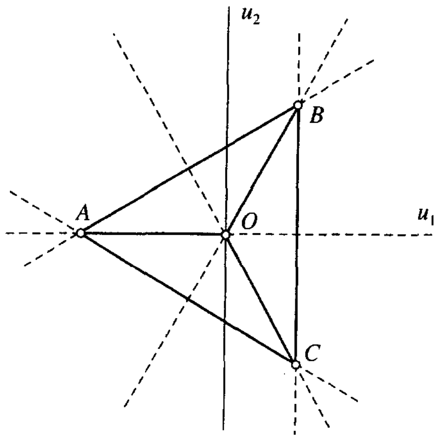

Fig. 1

obtaining, in terms of the new variables, a universal form of the equations of the ICs, independent of κ and g (the prime will be omitted from now on):

$$\frac{du_2}{du_1} = \frac{u_1^2 - u_2^2 - 2u_1 \pm Q_1}{2u_2(u_1 - 1)}$$
$$Q_1 = \sqrt{(u_1^2 - u_2^2 + 2u_1)^2 + 4u_2^2(u_1 - 1)^2} \tag{2.4}$$

The characteristic velocities in these variables are

$$\lambda_{1,2} = f - \kappa G^2\{2(u_1^2 + u_2^2) \pm Q_1\} \tag{2.5}$$

and, unlike the ICs, they behave differently for different signs of κ.

3. INTEGRAL CURVES IN THE PHASE PLANE u_1, u_2

The symmetry of the elastic potential mentioned in Section 1 makes formulae (2.4) for the ICs symmetrical relative to rotation through the angle $2\pi/3$ and relative to the u_1 axis, so that it will suffice to investigate the phase portrait of the ICs in the u_1, u_2 plane inside an angle $\pi/3$.

If $u_\alpha \gg 1$ (i.e. the anisotropy parameter g is very small), Eqs (2.4) define circles about the origin for fast waves and rays for small waves, as in an isotropic medium. In the general case, the curves (2.4) in the u_1, u_2 plane have four singular points with coordinates $O(0, 0)$, $A(-2, 0)$, B, $C(1, \pm\sqrt{3})$ (Fig. 1).

Let us follow the slopes of the integral curves (2.4) on straight lines passing through the singular points A, B and C. On the segment of the straight line $u_1 = 1$ in the range $|u_2| < \sqrt{3}$, that is, on the side BC of the triangle, we obtain $(du_2/du_1)_s = 0$ for slow waves (subscript s) and $(du_2/du_1)_f = \infty$ for fast waves (subscript f). Thus the side BC of the triangle is an integral curve of the family of fast waves, and, by symmetry, the same holds for the other sides, AB and AC. The ICs of the slow waves are orthogonal to the sides of the triangle. At the points where they intersect BC the functions $u_2 = u_2(u_1)$ representing the ICs have an extremum, namely, a minimum. By symmetry, the same is true for the ICs of the slow family at their intersections with the other sides of the triangle, AB and AC. In the domain $|u_2| > \sqrt{3}$, on the straight line $u_1 = 1$, on the contrary,

$$(du_2/du_1)_s = \infty, \quad (du_2/du_1)_f = 0$$

and the extension of the side BC (and the other sides as well) into the external domain (relative to the triangle ABC) serve as ICs of slow waves; the ICs of the fast family are orthogonal to them, and at their points of intersection with the straight line $u_1 = 1$, the functions $u_2 = u_2(u_1)$ representing them have a maximum.

It can be shown in an analogous fashion that the segment of the u_1 axis between the singular points $A(-2, 0)$ and $O(0, 0)$ is an IC of fast waves and all the rest of the u_1 axis is an IC of slow waves. By

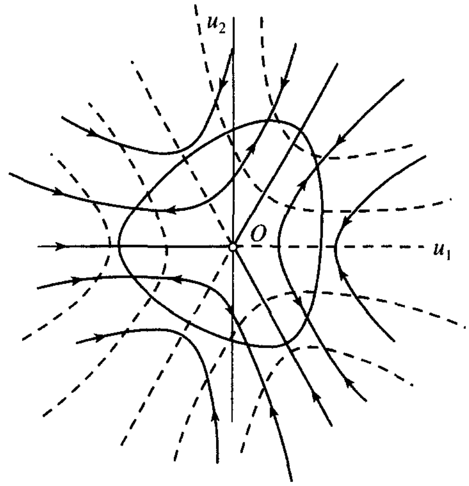

Fig. 2

symmetry, the other segments connecting the origin to the vertices of the triangle belong to the family of fast waves and their extensions to the family of slow waves. The solid lines in Fig. 1 represent the rectilinear ICs of the family of fast waves, and the dashed lines those of slow waves.

Let us consider the field of ICs around the singular points. In the neighbourhood of O we may assume that $u_\alpha \ll 1$ and replace Eq. (2.4) by the approximation

$$\frac{du_2}{du_1} = \frac{u_1 \pm \sqrt{u_1^2 + u_2^2}}{u_2} \tag{3.1}$$

Integration using polar coordinates r, θ ($u_1 = r\cos\theta$, $u_2 = r\sin\theta$) yields

$$r^3 = \frac{C}{(\cos\theta \mp 1)(\cos\theta \pm 1/2)} \tag{3.2}$$

where the upper (lower) sign is chosen for slow (fast) waves.

Thus, the ICs of the two families are mutually orthogonal and are almost hyperbolae with asymptotes along the rays $\theta = 0$, $\theta = \pm 2\pi/3$ for slow waves and $\theta = \pm\pi/3$, $\theta = -\pi$ for fast waves (Fig. 2).

Of the remaining singular points, it will suffice (by symmetry) to consider just one, say $A(-2, 0)$. Transform to a coordinate system x, y attached to that point, putting $u_1 = x - 2$, $u_2 = y$, and consider a small neighbourhood of the point A. The equations of the ICs (2.4) in the linear approximation are

$$dy/dx = (-x \mp \sqrt{x^2 + 9y^2})/(3y) \tag{3.3}$$

As always, the upper (lower) sign in the formula corresponds to slow (fast) waves. The ICs of both families may reach this singular point only in directions for which the radius vector coincides with the tangent to an IC, $y/x = dy/dx$. Hence, using Eq. (3.3), we obtain $y = 0$, that is, the u_1 axis, and $y = \pm\sqrt{3}x$, that is, the directions of the sides of the triangle AB and AC and their extensions beyond the vertex A. The ICs of fast waves, going along the u_1 axis, leave the singular point A and reach the other singular point O.

In the domain adjacent to the u_1 axis we may assume that $x_2 \gg 9y^2$. Then we obtain for slow (fast) waves in the neighbourhood of A:

$$y = \pm Cx^{3/2} \text{ in the domain } x < 0 \ (x > 0), \text{ i.e. for } u_1 < -2 (u_1 > -2),$$
$$x^2 + \frac{3}{2}y^2 = C \text{ in the domain } x > 0 \ (x < 0), \text{ i.e. for } u_1 > -2 (u_1 < -2),$$

where C is a constant of integration; the ICs reach the singular point as tangents to the u_1 axis. ICs $y = \pm Cx^{3/2}$ leaving the singular point A remain inside the angle between the rays emanating from A at angles $\pm\pi/6$ to the u_1 axis. Each IC of the fast family then reaches another singular point B or C, as tangent to the ray going from the origin to that point.

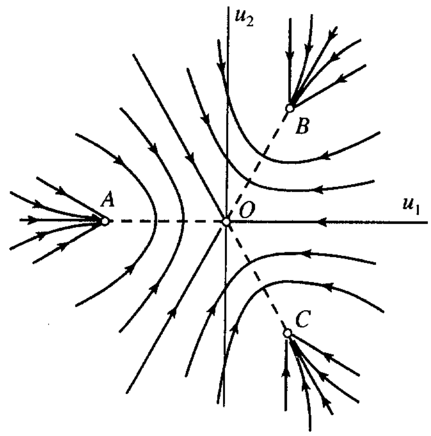

Fig. 3

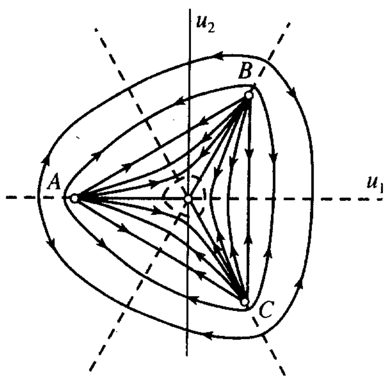

Fig. 4

The straight lines bounding the aforementioned angle are also ICs and pass through the singular points, forming the triangle ABC. The behaviour of the neighbouring ICs may be observed by taking as an example the neighbourhood of the straight line $u_1 = 1$. Set $u_1 - 1 = x$, where x is assumed to be small. For the fast waves we obtain $x = \pm C\sqrt{|3 - \upsilon^2|}$. On approaching the singular points, these curves become tangents to the rays going from the origin to the singular points, or adhere to arcs of ellipses going around the singular points. The overall phase portrait of the ICs in the u_1, u_2 plane is shown in Fig. 3 (slow waves) and Fig. 4 (fast waves).

4. THE CHANGE OF THE PERTURBATION PROFILE IN A RIEMANN WAVE

Since the characteristic velocities c_α (2.3) depend on the solution u_α, it follows that as a disturbance propagates, its form will change. If c_α increases as u_α varies along its IC, the wave has a tendency to reverse direction. We indicate (by the arrows in Figs 3 and 4) the directions of increasing c along ICs that lead to a reversal of waves. Instead of the characteristic velocities, we will use their squares $\lambda_\alpha = c_\alpha^2$. Denote the element of length on an IC by dl. We have to determine the sign of $d\lambda_\alpha/dl$ and plot the changes in its sign in the field of the IC in the u_1, u_2 plane. To fix our ideas, let us assume that the non-linearity parameter is positive, $\kappa > 0$; if $\kappa < 0$, the directions of the arrows in the figures should be reversed.

Instead of $d\lambda_\alpha/dl$, we can compute the quantity

$$\frac{d\lambda_\alpha}{du_1} = \frac{\partial\lambda_\alpha}{\partial u_1} + \frac{\partial\lambda_\alpha}{\partial u_2}\frac{du_2}{du_1}$$

along an IC, using formulae (2.4) and (2.5). We have

$$\frac{d\lambda_{1,2}}{du_1} = \frac{-\kappa}{q(u_1-1)}\{(q(3u_1^2+3u_2^2-14u_1+2)\pm$$
$$\pm(3(u_1^2+u_2^2)^2+8(2u_1^3+2u_1^2+2u_2^2-u_1-5u_1u_2^2)))\} \tag{4.1}$$
$$q = \sqrt{(u_2^2-u_1^2)^2+4u_2^2(u_1-1)^2}$$

These formulae are suitable everywhere, except in domains where the direction of the ICs is nearly parallel to the u_2 axis. There formulae (4.1) must be replaced by similar formulae for

$$\frac{d\lambda_\alpha}{du_2} = \frac{\partial\lambda_\alpha}{\partial u_2}+\frac{\partial\lambda_\alpha}{\partial u_1}\frac{du_1}{du_2}$$

We will first indicate the direction in which λ_1 increases on an IC of the family of slow waves. Along the u_2 axis, that is, for $u_1 = 0$, we have $d\lambda_1/du_1 > 0$ for all ICs of the slow family. For $u_2 = 0$ (along the u_1 axis, which for $u_1 < -2$ and $u_1 > 0$ is itself an IC)

$$q = |u_1(u_1+2)|d\lambda_1/du_1 = -2\kappa(1+3u_1)$$

that is, $d\lambda_1/du_1 < 0$ for $u_1 > 0$ and $d\lambda_1/du_1 > 0$ for $0 > u_1 > -2$. In the range $-2 < u_1 < 0$, where the ICs are orthogonal to the u_2 axis,

$$d\lambda_1/dl = d\lambda_1/du_2 = -Bu_2$$

where $B > 0$ is a coefficient. Thus, the segment $-2 < u_1 < 0$ is a curve across which $d\lambda_1/dl$ changes sign; in the upper half-plane, $d\lambda_1/dl < 0$, and in the lower half-plane, $d\lambda_1/dl > 0$. By symmetry, the other segments of the rays OB and OC are also lines across which $d\lambda_1/dl$ changes sign.

On a line connecting singular points, such as B and C, we obtain for $u_1 = 1$, $|u_2| < \sqrt{3}$

$$d\lambda_1/du_1 = -8\kappa<0$$

that is, on no side of the triangle ABC does $d\lambda_1/dl$ change sign. On the extensions of the sides beyond the vertices of the triangle ($|u_2| > \sqrt{3}$)

$$d\lambda_1/du_2 = -6\kappa u_2$$

that is, $d\lambda_1/du_2 < 0$ above B and $d\lambda_1/du_2 > 0$ below C. Thus, λ_1 increases along its ICs in the direction of the origin. A change in the direction of increase occurs across segments of the rays connecting the origin to the singular points.

For the family of fast waves, at points where their ICs intersect straight lines passing through the singular points A, B and C, say the straight line $u_1 = 1$, we obtain in the domain $|u_2| > \sqrt{3}$

$$d\lambda_2/du_1 = -20\kappa<0$$

The same straight line $u_1 = 1$ with $|u_2| < \sqrt{3}$ serves as an IC of the family, and on it

$$d\lambda_2/dl = d\lambda_2/du_2 = -6\kappa u_2$$

which means that across the u_1 axis the derivative $d\lambda_2/dl$ changes sign, from negative in the upper half-plane to positive in the lower half-plane. In the neighbourhood of the entire abscissa axis, except for the segment $-2 < u_1 < 0$,

$$d\lambda_2/dl = d\lambda_2/du_2 = Bu_2(1-4u_1), \quad B>0$$

Thus, the entire u_1 axis, with the exception of the segment indicated (and together with it all rays to singular points), is a curve across which the direction in which λ_2 increases is switched. The sign of $d\lambda_2/dl$ changes from minus in the upper half-plane to plus in the lower half-plane on the segment $u_1 > 1/4$ of the axis, and conversely on the other segments $0 < u_1 < 1/4$ and $u_1 < -2$. In the interval $-2, 0$ of the abscissa axis, which runs along an IC,

$$d\lambda_2/dl = -2\kappa(3u_1 + 1)$$

and the change of direction of increasing λ_2 occurs at the point $u_1 = -1/3$.

Thus, the lines across which $d\lambda_2/dl$ changes sign are the medians of the triangle ABC from the singular points and their extensions beyond the triangle, with the exception of the segments from the vertices to the origin. In addition, there is one more sign-change curve in the neighbourhood of the origin – a closed curve around O, intersecting the medians at a distance $1/3$ on the side of the vertex and $1/4$ on the side of the base. The equation of that curve may be written, for example, as $d\lambda_2/du_1 = 0$. The directions in which the characteristic velocities $c_1^2 = \lambda_1$ and $c_2^2 = \lambda_2$ increase are shown by arrows for the slow and fast Riemann waves, respectively, in Figs 3 and 4 for media with $\kappa > 0$. The dashed lines are those across which $d\lambda_\alpha/dl$ changes sign. On an increased scale, the closed curve across which the velocities of fast waves change sign is shown in Fig. 2 by a thin contour. The arrows indicate the direction in which the characteristic velocity increases.

If $\kappa < 0$, the upper sign everywhere in the formulae for $c_{1,2}^2$ and du_2/du_1 is assigned to the fast waves, and the lower sign to the slow waves. The form of the ICs in the u_2, u_1 plane remains as before, except that Fig. 3 is the phase portrait of the fast waves and Fig. 4 that of the slow waves, and the directions of the arrows should be reversed.

We recall that the whole portrait in the u_1, u_2 plane was constructed in normalized variables u'_α/G, where G is a small quantity. Returning to the physical variables, one sees that the singular points A, B an C lie at a small distance $2\sqrt{G}$ from the origin, and all the complexity of the behaviour of the ICs appears in the domain of small u_1, u_2.

5. RIEMANN WAVES IN AN ELASTIC CUBIC CRYSTAL

The complex pattern of the behaviour of the ICs and the large number of singular points is due to the presence of a slight anisotropy in the non-linear elastic medium. But if the anisotropy is not small and the coefficient g of the cubic powers in formula (1.4) is finite, the non-linear term with κ may be ignored. At $\kappa = 0$ we obtain the problem of low-amplitude non-linear waves propagating along the principal diagonal in a cubic crystal. The expressions for the characteristic velocities and the equations of the ICs are

$$c_{1,2}^2 = \lambda_{1,2} = f \mp 2g\sqrt{u_1^2 + u_2^2}$$

$$\frac{du_2}{du_1} = \frac{u_1 \pm \sqrt{u_1^2 + u_2^2}}{u_2}$$

Such ICs were represented above in the neighbourhood of the singular point $O(0, 0)$ by Eq. (3.1), as shown in Fig. 2 in the neighbourhood of the origin. Now, however, this pattern acts throughout the u_1, u_2 plane. The quantities λ_1 and λ_2 vary in the same way as inside the closed curve of sign change around the origin (Figs 3 and 4).

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